

# CALIBRATED ASSOCIATIVE AND CAYLEY EMBEDDINGS

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**ABSTRACT.** Using the Cartan-Kähler theory, and results on real algebraic structures, we prove two embedding theorems. First, the interior of a smooth, compact 3-manifold may be isometrically embedded into a  $G_2$ -manifold as an associative submanifold. Second, the interior of a smooth, compact 4-manifold  $K$ , whose double  $doub(K)$  has a trivial bundle of self-dual 2-forms, may be isometrically embedded into a  $Spin(7)$ -manifold as a Cayley submanifold. Along the way, we also show that Bochner's Theorem on real analytic approximation of smooth differential forms, can be obtained using real algebraic tools developed by Akbulut and King.

## 1. INTRODUCTION

Let  $(M^7, g)$  be a Riemannian 7-manifold whose holonomy group  $\text{Hol}(g)$  is a subgroup of the exceptional group  $G_2$ . Then  $M$  is naturally equipped with a covariantly constant 3-form  $\varphi$  and 4-form  $*\varphi$ . We call  $(M, \varphi, g)$  a  $G_2$ -manifold. It is well known that  $\varphi$  and  $*\varphi$  are *calibrations* on  $M$ , in the sense of Harvey and Lawson [12]. The corresponding calibrated submanifolds in  $M$  are called *associative 3-folds* and *coassociative 4-folds*, respectively.

Similarly, if  $(M^8, g)$  has  $\text{Hol}(g) \subseteq Spin(7)$ , then  $M$  admits a covariantly constant, self-dual 4-form  $\Psi$ , and we call  $(M, \Psi, g)$  a  $Spin(7)$ -manifold. The 4-form  $\Psi$  is the *Cayley calibration*, and the calibrated submanifolds are *Cayley 4-folds*.

Constructing examples of manifolds with  $G_2$  and  $Spin(7)$  holonomy and their calibrated submanifolds is of interest because of their importance in string theory. Also, they provide new examples of volume minimizing submanifolds in a given homology class [12]. In [8], R. Bryant applied the Cartan-Kähler theory to show that: (1) every closed, real analytic, oriented Riemannian 3-fold can be isometrically embedded in a Calabi-Yau 3-fold as a special Lagrangian submanifold; and (2) every closed, real analytic, oriented Riemannian 4-fold with a trivial bundle of self-dual 2-forms can be isometrically embedded in a  $G_2$ -manifold as an coassociative submanifold. Moreover, the submanifolds above may be embedded as the fixed locus of a real structure (in the special Lagrangian case), or an anti  $G_2$ -involution (in the coassociative case).

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In this paper, we will first show that Bryant's constructions can be repeated for the associative and Cayley submanifolds.

**Theorem 1.1.** *Assume  $(K^3, g)$  is a closed, oriented, real analytic Riemannian 3-manifold. Then there exists a  $G_2$ -manifold  $(N^7, \varphi)$  and an isometric embedding  $i : K \hookrightarrow N$  such that the image  $i(K)$  is an associative submanifold of  $N$ . Moreover,  $(N, \varphi)$  can be chosen so that  $i(K)$  is the fixed point set of a nontrivial  $G_2$ -involution  $r : N \rightarrow N$ .*

*Remark.* Bryant showed that  $K$  isometrically embeds as a special Lagrangian submanifold of a Calabi-Yau 3-fold  $CY$ . This immediately yields an elementary version of Theorem 1.1 as  $N = CY \times \mathbb{R}$  naturally carries a  $G_2$  structure such that  $A$  isometrically embeds as an associative submanifold. However,  $CY \times \mathbb{R}$  has holonomy a subgroup of  $SU(3)$ . It may be checked that, as long as  $A$  is not flat, the  $N$  of Theorem 1.1 has holonomy exactly  $G_2$ . In particular, these  $N$  are not of the form  $CY \times \mathbb{R}$ . See the remark at the end of §6.

**Theorem 1.2.** *Assume  $(K^4, g)$  is a closed, oriented, real analytic Riemannian 4-manifold with a trivial bundle of self-dual 2-forms. Then there exists a  $Spin(7)$ -manifold  $(N^8, \Psi)$  and an isometric embedding  $i : K \hookrightarrow N$  whose image is a Cayley submanifold in  $N$ . Moreover,  $(N, \Psi)$  can be chosen so that  $i(K)$  is the fixed locus of a nontrivial  $Spin(7)$ -involution  $r : N \rightarrow N$ .*

We refer the reader to [8, §0.4] for a discussion of Cartan-Kähler theory that will be used in the constructions.

Making use of the real analytic implicit function theorem and a theorem of Nash-Tognoli we are able to show that Theorems 1.1 and 1.2 extend to interiors of compact, smooth manifolds. In particular, assume that  $K$  is a compact, oriented, smooth manifold, possibly with boundary. Let  $doub(K)$  denote the *doubling* of  $K$ : glue two copies of  $K$  together along the boundary with the identity map. If  $K$  is closed ( $\partial K = \emptyset$ ) then  $doub(K) = K$ . The manifold  $doub(K)$  is closed and orientable, and admits the structure of a real analytic Riemannian manifold; see Lemma 5.5. Then we have the following two corollaries.

**Theorem 1.3.** *Let  $A$  be the interior of a smooth, orientable, compact 3-manifold  $K$  with nonempty boundary. Then  $A$  admits a compatible real analytic Riemannian structure. There exists a  $G_2$ -manifold  $(N^7, \varphi)$  and an isometric embedding  $i : A \hookrightarrow N$  such that  $i(A)$  is an associative submanifold in  $N$ . Moreover  $(N, \varphi)$  may be chosen so that  $i(A)$  is the fixed locus of a nontrivial  $G_2$ -involution  $r : N \rightarrow N$ .*

**Theorem 1.4.** *Let  $A$  be the interior of a smooth, orientable, compact 4-manifold  $K$  with nonempty boundary. Then  $A$  admits a compatible real analytic Riemannian structure. Assume also that the bundle of self-dual 2-forms over  $doub(K)$  is trivial. There exists a  $Spin(7)$ -manifold  $(N^8, \Psi)$  and an isometric embedding  $i : A \hookrightarrow N$  whose image  $i(A)$  is a Cayley submanifold in*

$N$ . Moreover,  $(N, \Psi)$  may be chosen so that  $i(A)$  is the fixed point set of a nontrivial  $\text{Spin}(7)$ -involution  $r : N \rightarrow N$ .

Theorems 1.1 & 1.3 and Theorems 1.2 & 1.4 are proven in §6 and 7, respectively. Also note that in all these theorems  $N$  does not have to be a (locally) product manifold.

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## 2. ASSOCIATIVE SUBMANIFOLDS OF $G_2$ -MANIFOLDS

**2.1.  $G_2$ -manifolds and the associative calibration.** On the imaginary octonians  $\mathbb{R}^7 = \text{Im}(\mathbb{O})$  let  $x = (x^j)$  denote the standard linear coordinates, set  $dx^{jk} := dx^j \wedge dx^k$ , and define the 3-forms  $dx^{jk\ell} := dx^j \wedge dx^k \wedge dx^\ell$  and  $\varphi_0 := dx^{123} + dx^1 \wedge (dx^{45} + dx^{67}) + dx^2 \wedge (dx^{46} - dx^{57}) + dx^3 \wedge (-dx^{47} - dx^{56})$ .

The simple Lie group  $G_2$  is the subgroup of  $\text{GL}(7)$  preserving  $\varphi_0$  [7].

A  $G_2$ -structure on  $M^7$  is a principle right  $G_2$ -bundle  $\pi : P \rightarrow M$ . The elements of  $P_x = \pi^{-1}(x)$  are linear isomorphisms  $u : T_x M \rightarrow \mathbb{R}^7$ , and the right action is given by  $u \cdot a = a^{-1} \circ u$ . The  $G_2$ -structure induces a well-defined 3-form  $\varphi$  on  $M$  via  $\varphi_x = u^* \varphi_0$ . Additionally,  $M$  admits a unique metric  $g$  and volume form  $*1$  (also obtained by pull-back) for which  $u : T_x M \rightarrow \mathbb{R}^7$  is an oriented isometry. In particular,  $(*\varphi)_x = u^*(*\varphi_0)$ .

We say that  $(M, \varphi)$  is a  $G_2$ -manifold when  $\varphi$  and  $*\varphi$  are closed. Equivalently, the  $G_2$ -structure is torsion-free [11]. In this case,  $\varphi$  is parallel,  $M$  is Ricci-flat [4, 10.64], and the metric is real analytic in harmonic coordinates [10, Th. 5.2]. Since  $\varphi$  is harmonic it follows that  $\varphi$  is real analytic as well.

Assume  $(M, \varphi)$  is a  $G_2$ -manifold. Then  $\varphi$  is the *associative calibration*. The 3-dimensional submanifolds  $i : X^3 \hookrightarrow M$  calibrated by  $\varphi$  are the *associative submanifolds*. Associative submanifolds are plentiful in  $G_2$ -manifolds: it is a consequence of the Cartan-Kähler theorem [8, §0.4] that every associative  $E^3 \subset T_z M$  is tangent to an associative  $X^3 \subset M$ . Moreover,

**Lemma 2.1.** *Every real-analytic 2-dimensional submanifold  $Y^2$  of a  $G_2$ -manifold  $(M^7, \varphi)$  lies in a unique associative  $X^3$ .*

The flat case  $(M, \varphi) = (\mathbb{R}^7, \varphi_0)$  was proven by Harvey and Lawson [12, Th.4.1]. Given Lemma 2.2 below, the proof (at the end of this section) is a simple application of the Cartan-Kähler theorem [8, §0.4].

The fundamental identity [12, Th.1.6]

$$\varphi(u, v, w)^2 + |\chi(u, v, w)|^2 = |u \wedge v \wedge w|^2$$

implies that  $i^* \varphi = \text{dvol}$  precisely when  $i^* \chi = 0$ . Here  $\chi$  is the vector-valued 3-form defined by

$$\langle \chi(u, v, w), z \rangle = *\varphi(u, v, w, z).$$

In particular, the associative submanifolds are the 3-dimensional integral manifolds of  $\{\chi = 0\}$ . In the flat case  $(\mathbb{R}^7, \varphi_0)$ ,

$$*\varphi_0 = dx^{4567} + dx^{23} \wedge (dx^{45} + dx^{67}) + dx^{31} \wedge (dx^{46} - dx^{57}) + dx^{12} \wedge (-dx^{47} - dx^{56}),$$

and

$$\begin{aligned} \chi_0 = & - (dx^{357} - dx^{346} - dx^{256} - dx^{247}) \partial_{x^1} \\ & - (dx^{367} + dx^{345} + dx^{156} + dx^{147}) \partial_{x^2} \\ & + (dx^{267} + dx^{245} + dx^{157} - dx^{146}) \partial_{x^3} \\ & - (dx^{567} + dx^{235} - dx^{136} - dx^{127}) \partial_{x^4} \\ & + (dx^{467} + dx^{234} - dx^{137} + dx^{126}) \partial_{x^5} \\ & - (dx^{457} + dx^{237} + dx^{134} + dx^{125}) \partial_{x^6} \\ & + (dx^{456} + dx^{236} + dx^{135} - dx^{124}) \partial_{x^7}. \end{aligned}$$

Notice that the coefficient 3-forms are  $\chi_{0,j} := -\partial_{x^j} \lrcorner (*\varphi_0)$ .

Given an arbitrary  $G_2$  manifold  $(M^7, \varphi)$ , let  $\{\omega^1, \dots, \omega^7\}$  be a local  $G_2$  coframing. That is,  $\omega_x^j = u_x^* dx^j$  for smoothly varying isometries  $u_x : T_x M \rightarrow \mathbb{R}^7$  in  $P_x$ . Let  $\{e_j\}$  denote the dual framing. Then local expressions for  $\varphi = u^* \varphi_0$ ,  $*\varphi = u^*(*\varphi_0)$  and  $\chi = u^* \chi_0$  are given by replacing the terms  $dx$  and  $\partial_x$  in  $\varphi_0$ ,  $*\varphi_0$  and  $\chi_0$  with  $\omega$  and  $e$ , respectively. The associative submanifolds of  $(M, \varphi)$  are the 3-dimensional integral manifolds of  $\{\chi_j := -e_j \lrcorner (*\varphi)\}$ .

**Lemma 2.2.** *Let  $\mathcal{I}$  be the ideal algebraically generated by the coefficient 3-forms  $\chi_j$ . Then  $\mathcal{I}$  is well-defined and closed under exterior differentiation ( $d\mathcal{I} \subset \mathcal{I}$ ).*

*Proof.* That  $\mathcal{I}$  is well-defined (i.e. does not depend on choice of local  $G_2$ -coframing  $u_x$ ) is immediate from the  $G_2$ -invariance of the forms  $\varphi$ ,  $*\varphi$  and  $\chi$ . To see that  $\mathcal{I}$  is differentially closed recollect that the Lie derivative of any form  $\alpha$  by a vector field  $X$  is  $\mathcal{L}_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha)$ , so that

$$\begin{aligned} d\chi_j &= -d(e_j \lrcorner *\varphi) \\ &= e_j \lrcorner d(*\varphi) - \mathcal{L}_{e_j} *\varphi \\ &= -\mathcal{L}_{e_j} *\varphi, & (*\varphi \text{ is closed}), \\ &= (e_j \lrcorner d\omega^k) \wedge \chi_k. \end{aligned}$$

The last line follows from an application of [6, §V.8, Ex.8].  $\square$

*Proof of Lemma 2.1.* Since  $Y$  is 2-dimensional and  $\mathcal{I}$  is generated by 3-forms,  $Y$  is a priori an integral manifold. In order to apply the Cartan-Kähler theorem we must show that: (i)  $Y$  is regular; and (ii) the polar space  $H(T_y Y)$  is of dimension 3 for every  $y \in Y$ . (See [8, §0.4] for a review of polar spaces, the variety of  $p$ -dimensional integral elements  $V_p(\mathcal{I})_y$  in the Grassmannian  $\text{Gr}(p, T_y M)$  and the Cartan-Kähler theorem in this context.)

Regularity is easily confirmed, and in the course of doing so we will see that the polar space is of dimension three. Fix  $y \in Y$ . Since  $\mathcal{I}$  is generated by 3-forms,  $V_2(\mathcal{I})_y = \text{Gr}(2, T_y M)$ , and  $V_2(\mathcal{I})$  is (trivially) a smooth submanifold of  $\text{Gr}(2, TM)$  near  $T_y Y$ . Whence,  $T_y Y$  is ordinary.

Because  $G_2$  acts transitively on 2-planes, there is no loss of generality in assuming that  $T_y Y$  is spanned by  $\{e_1, e_2\}$ . The polar space of  $T_y Y$  is  $H(T_y Y) = \{v \in T_y M \mid \psi(v, e_1, e_2) = 0 \ \forall \ \psi \in \mathcal{I}^3\}$ . In our case it is straightforward to see that  $H(T_y Y) = \{v \in T_y M \mid \chi_j(v, e_1, e_2) = 0 \ \forall \ j\}$  is spanned by  $\{e_1, e_2, e_3\}$ . Whence the extension rank  $r(T_y Y) = \dim H(T_y Y) - (2 + 1) = 0$  is constant function  $V_2(\mathcal{I}) = \text{Gr}(2, TM)$ , and  $Y$  is regular. The result follows from the Cartan-Kähler theory.  $\square$

**2.2.  $G_2$ -involutions.** One way of finding examples of associative submanifolds is to investigate the fixed point sets of  $G_2$  involutions, [14, Prop.10.8.1].

Let  $\sigma : M \rightarrow M$  be a nontrivial isometric involution of a  $G_2$ -manifold  $(M, g)$ . This means that  $\sigma : M \rightarrow M$  is a diffeomorphism satisfying  $\sigma^*(g) = g$  and  $\sigma^2 = \text{id}$ , but  $\sigma \neq 1$ .

**Lemma 2.3.** *Let  $(M, \varphi, g)$  be a  $G_2$ -manifold and let  $\sigma : M \rightarrow M$  be a non-trivial isometric involution preserving  $\varphi$ , i.e.  $\sigma^*(\varphi) = \varphi$ . Then the fixed point set  $A = \{p \in M \mid \sigma(p) = p\}$  is an associative 3-fold in  $M$ .*

For the details of the proof, see [14]. Note that the fixed point set  $A$  is a closed submanifold of  $M$ . This is because  $A$  can be represented as a preimage of 0 under a continuous map,  $\sigma - \text{Id}$ . This does not contradict our assumption that  $A$  can be open. The reason for this is when we thicken an open manifold  $A$  to obtain the  $G_2$ -manifold  $M$ , then  $A$  (even if it is open in  $K$ ) will be always a closed submanifold of  $M$ .

Note that there is a similar construction for the coassociative case [14, Prop.10.8.5].

### 3. CAYLEY SUBMANIFOLDS OF $Spin(7)$ -MANIFOLDS

**3.1.  $Spin(7)$ -manifolds and the Cayley calibration.** The octonians  $\mathbb{O} = \mathbb{R}^8$  are equipped with a triple (and quadruple) cross product. This cross product defines a 4-form  $\Psi_0(u, v, w, z) = \langle u \times v \times w, z \rangle$ . Given linear coordinates  $x = (x^0, x^1, \dots, x^7)$  on  $\mathbb{R}^8$ ,

$$\begin{aligned} \widehat{\Psi}_0 &= dx^0 \wedge \phi_0 + * \phi_0 \\ &= dx^{0123} + dx^{4567} + (dx^{01} + dx^{23}) \wedge (dx^{45} + dx^{67}) \\ &\quad + (dx^{02} + dx^{31}) \wedge (dx^{46} + dx^{75}) + (dx^{03} + dx^{12}) \wedge (dx^{74} + dx^{65}) . \end{aligned}$$

The exceptional  $Spin(7) \subset SO(8)$  is the subgroup preserving  $\Psi_0$  [7, 12].

A  $Spin(7)$ -structure on  $M^8$  is a principle right  $Spin(7)$ -bundle  $\pi : P \rightarrow M$ . The elements of  $P_x = \pi^{-1}(x)$  are linear isomorphisms  $u : T_x M \rightarrow \mathbb{R}^8$ , and the right action is given by  $u \cdot a = a^{-1} \circ u$ . The  $Spin(7)$ -structure induces a well-defined 4-form  $\Psi$  on  $M$  via  $\Psi_x = u^* \Psi_0$ . As in the  $G_2$  case,  $M$  also admits a unique metric  $g$  and volume form  $*1$  for which  $u : T_x M \rightarrow \mathbb{R}^8$  is an oriented isometry.

We say  $M$  is a  $Spin(7)$ -manifold when  $\Psi$  is closed. (Equivalently,  $\Psi$  is parallel and the  $Spin(7)$ -structure is torsion-free [7].) In this case,  $M$  is Ricci-flat [4, 10.65], and the metric is real analytic in harmonic coordinates [10, Th. 5.2]. Since  $\Psi$  is harmonic it follows that  $\Psi$  is real analytic as well.

Given a  $Spin(7)$ -manifold, the 4-form  $\Psi$  is a calibration, known as the *Cayley calibration*. The 4-dimensional submanifolds  $i : X^4 \rightarrow M$  calibrated by  $\Psi$  are the *Cayley submanifolds*. Cayley submanifolds are plentiful in  $Spin(7)$ -manifolds: The Cartan-Kähler theory implies that given a Cayley plane  $E^4 \subset T_z M$ , there exists a Cayley submanifold  $X^4$  tangent to  $E$ . Moreover,

**Lemma 3.1.** *Every real-analytic 3-dimensional submanifold  $Y^3$  of a  $Spin(7)$ -manifold  $(M^8, \Psi)$  lies in a unique Cayley  $X^4$ .*

The flat case  $(M, \Psi) = (\mathbb{R}^8, \Psi_0)$  was proven by Harvey and Lawson [12, Th.4.3]. The proof follows from Lemma 3.3 below and the Cartan-Kähler Theorem [8, §0.4], and is given at the end of this section.

The 4-form satisfies the relation [12, Ch.5, Th.1.28]

$$(3.2) \quad \Psi_0(u, v, w, z)^2 + |\operatorname{Im}(u \times v \times w \times z)|^2 = |u \wedge v \wedge w \wedge z|^2.$$

Let  $\tau_0$  denote the vector-valued 4-form  $\tau_0(u, v, w, z) = \operatorname{Im}(u, v, w, z)$ . Given a  $Spin(7)$ -manifold  $(M^8, \Psi)$  and a submanifold  $i : X^4 \hookrightarrow M$ , notice that  $i^* \Psi = \operatorname{dvol}_X$  if and only if  $i^* \tau = 0$ , where  $\tau$  is the vector valued 4-form  $\tau_x = u^* \tau_0$ . Consequently the Cayley submanifolds are the integral submanifolds of  $\tau = 0$ . Let  $\{\omega^0, \dots, \omega^7\}$  be a local  $Spin(7)$ -coframing. That is,  $\omega_x^j = u_x^* dx^j$ ,  $j = 0, \dots, 7$ , for smoothly varying isometries  $u_x : T_x M \rightarrow \mathbb{R}^8$  in  $P_x$ . Let  $\{e_j\}$  denote the dual framing. Write  $\tau = \sum_1^7 \tau^j e_j$ . Then  $X$  is Cayley if and only if  $i^* \tau^j = 0$ ,  $j = 1, \dots, 7$ .

**Lemma 3.3.** *Let  $\mathcal{I}$  denote the ideal generated algebraically by the  $\tau^j$ . Then  $\mathcal{I}$  is well-defined and closed under exterior differentiation.*

*Proof.* That  $\mathcal{I}$  is well-defined is a consequence of the  $Spin(7)$  invariance of  $\tau$ .

To see that  $\mathcal{I}$  is closed, note that (3.2) and the fact that  $\Psi$  is parallel imply that  $\tau$  is also parallel. Hence

$$\begin{aligned} 0 &= \nabla \tau = \nabla (\tau^j \otimes e_j) = \nabla \tau^j \otimes e_j + \tau^j \otimes \nabla e_j = \nabla \tau^j \otimes e_j - \tau^j \otimes \theta_k^j e_k \\ \implies \nabla \tau^j &= \tau^k \otimes \theta_k^j. \end{aligned}$$

Above,  $\theta$  is the  $\mathfrak{spin}(7)$ -valued connection form. Since the exterior derivative  $d\tau^j$  is the skew-symmetrization of the covariant derivative  $\nabla\tau^j$ , we have  $d\tau^j = \tau^k \wedge \theta_k^j$ .  $\square$

*Proof of Lemma 3.1.* As in the case of Lemma 2.1, the proof is a straightforward application of the Cartan-Kähler Theorem. See [8, §0.4] for a review of integral elements, polar spaces and the Cartan-Kähler theory in the context. As  $\mathcal{I}$  is generated by 4-forms, and  $Y$  is of dimension three,  $Y$  is trivially an integral manifold. Similarly,  $V_3(\mathcal{I}) = \text{Gr}_3(TM)$ , and  $T_y Y$  is ordinary, for all  $y \in Y$ .

In a  $Spin(7)$  coframing the  $\tau^j$  are given by

$$\begin{aligned} \tau^1 &= (\omega^{03} - \omega^{12}) \wedge (\omega^{46} + \omega^{57}) - (\omega^{02} + \omega^{13}) \wedge (\omega^{47} - \omega^{56}) \\ \tau^2 &= (\omega^{01} - \omega^{23}) \wedge (\omega^{47} - \omega^{56}) - (\omega^{03} - \omega^{12}) \wedge (\omega^{45} - \omega^{67}) \\ \tau^3 &= (\omega^{02} + \omega^{13}) \wedge (\omega^{45} - \omega^{67}) - (\omega^{01} - \omega^{23}) \wedge (\omega^{46} + \omega^{57}) \\ \tau^4 &= \omega^{1234} - \omega^{0235} + \omega^{0136} - \omega^{0127} + \omega^{0567} - \omega^{1467} + \omega^{2457} - \omega^{3456} \\ \tau^5 &= \omega^{1235} + \omega^{0234} + \omega^{0137} + \omega^{0126} - \omega^{1567} - \omega^{0467} - \omega^{3457} - \omega^{2456} \\ \tau^6 &= \omega^{1236} + \omega^{0237} - \omega^{0134} - \omega^{0125} - \omega^{2567} - \omega^{3467} + \omega^{0457} + \omega^{1456} \\ \tau^7 &= \omega^{1237} - \omega^{0236} - \omega^{0135} + \omega^{0124} - \omega^{3567} + \omega^{2467} + \omega^{1457} - \omega^{0456} . \end{aligned}$$

Cf. [18, (6.10)].

Fix  $y$ . Since  $Spin(7)$  acts transitively on 3-planes [7, Th.4], we may assume that  $T_y Y = \text{span}\{e_0, e_1, e_2\}$ . Then  $H(T_y Y) = \text{span}\{e_0, e_1, e_2, e_3\}$ , and the extension rank is zero. It follows from the Cartan-Kähler theorem that  $Y$  lies in a unique Cayley 4-manifold.  $\square$

**3.2.  $Spin(7)$ -involutions.** There are examples of Cayley submanifolds which are the fixed point sets of  $Spin(7)$  involutions, [14, Prop.10.8.6].

As in  $G_2$  case, let  $\sigma : M \rightarrow M$  be a nontrivial isometric involution of a  $Spin(7)$ -manifold  $(M, \Psi, g)$  satisfying  $\sigma^*(g) = g$  and  $\sigma^2 = id$ , but  $\sigma \neq 1$ .

**Lemma 3.4.** *Let  $(M, \Psi, g)$  be a  $Spin(7)$ -manifold and let  $\sigma : M \rightarrow M$  be a nontrivial isometric involution preserving  $\Psi$ , i.e.  $\sigma^*(\Psi) = \Psi$ . Then each connected component of the fixed point set  $A = \{p \in M \mid \sigma(p) = p\}$  is either a Cayley 4-fold in  $M$  or a single point.*

For the details of the proof, see [14].

#### 4. $G$ -STRUCTURES AND IDEALS

The primary purpose of this section is to introduce the principle objects of interest, and establish notation. Detailed discussions may be found in [15, Ch.II] and [7, §1].

**4.1.  $G$ -structures.** Given a smooth manifold  $M$  of dimension  $n$ , let  $\pi : \mathcal{F} \rightarrow M$  denote its bundle of  $\mathbb{R}^n$ -valued coframes. The fibre  $\pi^{-1}(x) =: \mathcal{F}_x$  over  $x \in M$  is the collection of linear isomorphisms  $u : T_x M \rightarrow \mathbb{R}^n$ . This is a principle right  $\mathrm{GL}(n)$ -bundle with action of  $a \in \mathrm{GL}(n)$  given by  $u \cdot a := a^{-1} \circ u$ . Given a subgroup  $G \subset \mathrm{GL}(n)$ , a  $G$ -structure is a principle sub-bundle  $\mathcal{P} \subset \mathcal{F}$  with structure group  $G$ .

*Example.* When  $G = \mathrm{SO}_n$ , there is a unique Riemannian metric on  $M$  for which  $\mathcal{P}$  is the bundle of (oriented) orthonormal coframes.

In the case that  $G \subset \mathrm{SO}(n)$ , let  $\overline{\mathcal{P}} := \mathcal{P} \cdot \mathrm{SO}(n)$  be the  $\mathrm{SO}(n)$ -bundle of orthonormal coframes. The corresponding Riemannian metric on  $M$  is the *underlying metric of the  $G$ -structure*.

**4.2. Flat structures.** Given a coordinate neighborhood  $x : U \rightarrow \mathbb{R}^n$  on  $M$ , notice that  $dx$  is a local section  $U \rightarrow \mathcal{F}$ . We say the  $G$ -structure  $\mathcal{P}$  is *flat* when  $M$  every  $p \in M$  admits a coordinate chart such that  $dx$  is a local section of  $\mathcal{P}$ .

Clearly,  $\mathcal{F}$  is a flat  $\mathrm{GL}(n)$ -structure. Every orientable  $M$  admits a  $\mathrm{SL}(n)$ -structure, given by the volume form. (Alternatively, every  $\mathrm{SL}(n)$ -structure on  $M$  uniquely determines a volume form.) Because the volume form may always be expressed locally as  $dx^1 \wedge \cdots \wedge dx^n$  in some local coordinate system, every  $\mathrm{SL}(n)$ -structure is flat.

**4.3. Connections.** Given a  $G$ -structure  $\pi : \mathcal{P} \rightarrow M$ , a tangent vector  $v \in T_u \mathcal{P}$  is *vertical* if  $\pi_*(v) = 0$ . A differential  $p$ -form  $\Omega$  on  $\mathcal{P}$  is *semi-basic* if  $v \lrcorner \Omega = 0$  for all vertical  $v$ . There is a canonically defined,  $\mathbb{R}^n$ -valued semi-basic 1-form  $\eta$  on  $\mathcal{P}$ : given  $v \in T_u \mathcal{P}$ ,

$$\eta(v) := u \circ \pi_*(v).$$

The components of  $\eta = (\eta^1, \dots, \eta^n)$  give a basis of the semi-basic 1-forms on  $\mathcal{P}$ .

Let  $V_u := \ker \pi_* \subset T_u \mathcal{P}$  denote the vertical subspace at  $u \in \mathcal{P}$ . A *connection* on  $\mathcal{P}$  is a smooth distribution  $H_u \subset T_u \mathcal{P}$  that is complimentary to  $V_u$  and invariant under the right action of  $G$ . Equivalently,  $\pi_* : H_u \rightarrow T_x M$  is an isomorphism,  $x = \pi(u)$ ; and  $(R_a)_* H_u = H_{u \cdot a}$ , where  $R_a : \mathcal{P} \rightarrow \mathcal{P}$  is the map  $u \mapsto u \cdot a$ .

The connection  $H$  determines a  $\mathfrak{g}$ -valued *connection 1-form*  $\theta$  satisfying  $(R_a)^* \theta = \mathrm{ad}(a^{-1}) \theta$  as follows. Setting  $\Theta|_{H_u} \equiv 0$ , it remains to specify  $\theta$  on  $V_u$ . Every  $X \in \mathfrak{g}$  determines a vertical vector field  $X^*$  on  $\mathcal{P}$ : given  $a(t) \in G$  with  $a(0) = \mathrm{Id}$  and  $a'(0) = X$ , define  $X_u^* = \frac{d}{dt} u \cdot a(t)|_{t=0}$ . These  $X_u^*$  span  $V_u$ , and defining  $\theta(X^*) = X$  determines  $\theta$ . Clearly the connection form is  $\mathfrak{g}$ -valued. We leave it to the reader to confirm that  $(R_a)^* \theta = \mathrm{ad}(a^{-1}) \theta$ . Conversely, any  $\mathfrak{g}$ -valued 1-form satisfying this condition determines a connection  $H$  via the assignment  $H_u := \{\theta_u = 0\}$ .



**4.4. Torsion.** It can be shown that a  $\mathfrak{g}$ -valued 1-form  $\theta$  is a connection form if and only if  $d\eta^j = -\theta_k^j \wedge \eta^k + T_{k\ell}^j \eta^k \wedge \eta^\ell$ , with  $T_{k\ell}^j + T_{\ell k}^j = 0$ , [13, Prop. 8.3.3]. The functions  $T_{k\ell}^j$  define a map  $T := T_{k\ell}^j \frac{\partial}{\partial x^j} \otimes dx^k \wedge dx^\ell : \mathcal{P} \rightarrow \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)^*$ , called the *torsion of  $\theta$* . Any other connection 1-form  $\tilde{\theta}$  differs from  $\theta$  by a  $\mathfrak{g}$ -valued semi-basic 1-form,  $\tilde{\theta}_k^j = \theta_k^j + c_{k\ell}^j \eta^\ell$ . The corresponding change in torsion is  $\tilde{T}_{k\ell}^j - T_{k\ell}^j = c_{k\ell}^j - c_{\ell k}^j$ . In particular,  $\tilde{T} - T$  takes values in the image of the skew-symmetrizing map  $\delta : \mathfrak{g} \otimes (\mathbb{R}^n)^* \subset \mathbb{R}^n \otimes (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \rightarrow \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)^*$ . This leads to the definition of the *torsion of the  $G$ -structure  $\mathcal{P}$*  as  $[T] : \mathcal{P} \rightarrow h^0(\mathfrak{g}) := (\mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)^*)/\delta(\mathfrak{g} \otimes (\mathbb{R}^n)^*)$ .

*Example.* When  $\mathfrak{g} = \mathfrak{so}(n)$ , it is easy to show that  $h^0(\mathfrak{so}(n)) = \{0\}$ . This is equivalent to the existence of a torsion-free  $g$ -compatible connection on a Riemannian manifold  $(M, g)$ . (This is the existence-half of the fundamental theorem of Riemannian geometry. The uniqueness-half is equivalent to  $\mathfrak{so}(n)^{(1)} := \ker \delta = (\mathfrak{so}(n) \otimes (\mathbb{R}^n)^*) \cap (\mathbb{R}^n \otimes S^2(\mathbb{R}^n)^*) = \{0\}$ . In general,  $\mathfrak{g}^{(1)}$  records the changes in connection that preserve torsion.) When  $\mathfrak{g} \subset \mathfrak{so}(n)$ , we have  $h^0(\mathfrak{g}) = (\mathfrak{so}(n)/\mathfrak{g}) \otimes (\mathbb{R}^n)^*$ .

**4.5. Torsion-free  $G$ -structures.** We say that  $\mathcal{P}$  is *torsion-free* when  $[T] \equiv 0$ , and  $M$  is a  *$G$ -manifold*. In the case that  $G \subset \mathrm{SO}(n)$ , let  $i : \mathcal{P} \rightarrow \overline{\mathcal{P}}$  denote the inclusion, and  $\overline{H}$  the Levi-Civita connection on  $\overline{\mathcal{P}}$ . It is not difficult to see that  $\mathcal{P}$  is torsion-free if and only if  $\overline{H} \subset i_* T\mathcal{P}$ . Equivalently,  $\mathcal{P}$  is preserved under parallel transport in  $\overline{\mathcal{P}}$ .

Torsion may be viewed as a first-order obstruction to flatness. Here is one way to see this. Suppose  $M$  carries a  $G$ -structure  $\mathcal{P}$ . Let  $x : U \rightarrow \mathbb{R}^n$  be a coordinate system about  $z \in M$ . We may assume that the local section  $dx : U \rightarrow \mathcal{F}$  satisfies  $dx_z \in \mathcal{P}_z$ . The coordinates define a local, flat  $G$ -structure  $\mathcal{P}_0 := dx \cdot G$  over  $U$ . The following lemma is well-known.

**Lemma 4.1.** *The  $G$ -structure  $\mathcal{P}$  is torsion-free if and only if for all  $z \in M$ , there exist local coordinates  $(x, U)$  such that  $\mathcal{P}$  and  $\mathcal{P}_0$  are tangent at  $dx_z$ .*

**4.6.  $G$ -structures as sections and 1-flatness.** Let  $S = \mathcal{F}/G$ , and consider the bundle  $\overline{\pi} : S \rightarrow M$ ,

$$\begin{array}{ccc} \mathcal{F} & & \\ \pi \downarrow & \searrow \rho & \\ & S & \\ & \swarrow \overline{\pi} & \\ & M & \end{array}$$

Notice that  $G$ -structures on  $M$  are in one-to-one correspondence with  $\overline{\pi}$ -sections  $\sigma : M \rightarrow S$ .

We say  $\sigma : M \rightarrow S$  is *flat* when the corresponding  $G$ -structure is flat. The section  $\sigma$  is *1-flat* if it is flat to first-order at every point. That is, if every  $x \in M$  admits an open neighborhood  $U$  carrying a flat  $G$ -structure with corresponding section  $\sigma_0 : U \rightarrow S$  such that  $\sigma(x) = \sigma_0(x)$ , and  $\sigma(M)$  and  $\sigma_0(U)$  are tangent at  $\sigma(x)$ . By Lemma 4.1,  $\mathcal{P}$  is *torsion-free* if and only if  $\sigma$  is *1-flat*.

**4.7. Admissible groups.** Given  $G \subset \mathrm{SO}(n)$ , let  $\Lambda^*(\mathbb{R}^n)^G$  denote the  $G$ -invariant constant coefficient differential forms on  $\mathbb{R}^n$ . We say  $G$  is *admissible* if it is the subgroup of  $\mathrm{GL}(n)$  fixing all the forms in  $\Lambda^*(\mathbb{R}^n)^G$ . (A priori, this subgroup contains  $G$ .) In [7] Bryant showed that  $G_2$  and  $\mathrm{Spin}(7)$  are admissible. The ring  $\Lambda^*(\mathbb{R}^n)^{G_2}$  is generated by  $\varphi_0$  and  $*\varphi_0$  (cf. §2.1), and  $\Lambda^*(\mathbb{R}^n)^{\mathrm{Spin}(7)}$  by  $\Psi_0$  (cf. §3.1).

**4.8. A differential ideal on  $S$ .** This subsection and the following borrow heavily from [8].

Every  $p$ -form  $\alpha$  on  $\mathbb{R}^n$  defines a semi-basic  $p$ -form  $\hat{\alpha}$  on  $\mathcal{F}$  via

$$\hat{\alpha}_u(v_1, \dots, v_p) := \alpha(\eta(v_1), \dots, \eta(v_p)).$$

When  $\alpha$  is  $G$ -invariant,  $\hat{\alpha}$  is invariant under the right-action of  $G$  on  $\mathcal{F}$  and therefore descends to a well-defined  $p$ -form on  $S$ , also denoted by  $\hat{\alpha}$ . Given a  $G$ -structure  $\mathcal{P}$  with corresponding section  $\sigma : M \rightarrow S$ , the pull-back  $\sigma^*\hat{\alpha}$  defines a  $p$ -form  $\alpha_\sigma$  on  $M$ . Recall that  $\sigma$  is torsion-free if and only if the  $G$ -structure is preserved under parallel transport by the underlying Levi-Civita connection. In particular,  $\alpha_\sigma$  is parallel, and therefore closed, if  $\sigma$  is torsion-free. Whence  $\sigma^*(d\hat{\alpha}) = 0$ .

We denote by  $\mathcal{I}$  both the ideal on  $\mathcal{F}$  and the ideal on  $S$  that is generated algebraically by  $d\hat{\alpha}$ ,  $\alpha \in \Lambda^*\mathbb{R}^n$ . Graphs of torsion-free  $\sigma : M \rightarrow S$  are necessarily integral manifolds of  $\mathcal{I}$ . The converse need not hold; see [8, §0.5.5] for an example.

**4.9. Strong admissibility.** Given  $k \leq n$ , let  $V(\mathcal{I}, \bar{\pi}) \subset \mathrm{Gr}_k(TS)$  denote the  $k$ -dimensional integral elements  $E \subset T_s S$  that are  $\bar{\pi}$ -transverse; that is, the projection  $\bar{\pi}_* : E \rightarrow T_{\bar{\pi}(s)} M$  is injective. As noted above,  $V_n(\mathcal{I}, \bar{\pi})$  contains the set of  $n$ -planes tangent to the graph of a torsion-free section  $\sigma$ . When  $G$  is admissible, and  $V_n(\mathcal{I}, \bar{\pi})$  consists of exactly these tangent planes, then we say  $G$  is *strongly admissible*. As a result any section  $\sigma : M \rightarrow S$  whose image in  $S$  is an integral manifold of  $\mathcal{I}$  is necessarily torsion-free. Both  $G_2$  and  $\mathrm{Spin}(7)$  are strongly admissible [7].

Recall from §4.4 that the torsion of a  $G$ -structure  $\sigma : M \rightarrow S$  lives in  $\mathfrak{h}^0(\mathfrak{g})$ . Since  $V_n(\mathcal{I}, \bar{\pi})$  contains the tangent planes to torsion-free  $\sigma(M)$ , and torsion is a first-order invariant, we must have

$$\mathrm{codim}(V_n(\mathcal{I}, \pi), \mathrm{Gr}_n(T\mathcal{F})) \leq \dim \mathfrak{h}^0(\mathfrak{g}),$$

with equality precisely when  $G$  is strongly admissible.

**4.10. Integral elements on  $\mathcal{S}$  and  $\mathcal{F}$ .** Define  $V_k(\mathcal{I}, \pi) \subset \text{Gr}_k(T_u\mathcal{F})$  to be the  $k$ -dimensional integral elements of  $\mathcal{I}$  that are  $\pi$ -transverse. Observe that  $\rho_* : V_k(\mathcal{I}, \pi) \rightarrow V_k(\mathcal{I}, \bar{\pi})$  is a surjection. Given  $E, E' \in V_k(\mathcal{I}, \pi)$ , we have  $\rho_*(E) = \rho_*(E')$  if and only if  $E \equiv E' \pmod{\mathfrak{g}_u}$ . Here  $\mathfrak{g}_u := \ker(\rho_*|_{T_u\mathcal{F}})$ . (Alternatively,  $\mathfrak{g}_u \subset V_u$  is the vertical subspace of  $T_u\mathcal{F}$  identified with  $\mathfrak{g}$  under the right action of  $G$  at  $u \in \mathcal{F}$ .) In particular, for fixed  $E$  the set of all such  $E'$  is naturally identified with  $\text{Hom}(E, \mathfrak{g}_u)$ . This set is of dimension  $k \dim(G)$ . As  $\dim T\mathcal{F} - \dim TS = n \dim(G)$ , we have

$$(4.2) \quad \text{codim}(V_n(\mathcal{I}, \bar{\pi}), \text{Gr}_n(TS)) = \text{codim}(V_n(\mathcal{I}, \pi), \text{Gr}_n(T\mathcal{F}))$$

in the case that  $k = n$ .

It is straightforward to check that, given  $E \in V_k(\mathcal{I}, \pi)$ , the polar spaces satisfy  $H(E) = (\rho_*)^{-1}H(\rho_*E)$ . In particular, given an integral flag  $F = \{E_j\}_{j=0}^n$  in  $T_u\mathcal{F}$  and the corresponding flag  $\bar{F} = \{\bar{E}_j = \rho_*E_j\}_{j=0}^n$  in  $T_{\rho(u)}S$ , we have

$$(4.3) \quad c_j(F) = c_j(\bar{F}) \quad \text{and} \quad c(F) = c(\bar{F}).$$

Finally, it is not difficult to describe the set  $V_n(\mathcal{I}, \pi) \subset \text{Gr}_n(T\mathcal{F})$ . Given a  $\pi$ -transverse  $E \in \text{Gr}_n(T\mathcal{F})$  the canonical 1-forms  $\eta^j$  span  $E^*$ . In particular, when restricted to  $E$  the connection 1-forms  $\theta_k^j$  may be expressed as linear combinations of the  $\eta^j$ ,  $\theta_k^j = p_{k\ell}^j \eta^\ell$ . The  $p_{k\ell}^j$  are functions on  $\mathcal{F}$ , and  $p_{k\ell}^j(u)$  parameterizes the open set of  $\pi$ -transverse  $n$ -planes  $E \in \text{Gr}_n(T_u\mathcal{F})$ . Now, given  $\alpha = a_I dx^I \in \Lambda^p(\mathbb{R}^n)^*$ ,  $I = \{i_1, \dots, i_p\}$  a multi-index, we have  $\hat{\alpha} = a_I \eta^I$ . When restricted to  $E \in \text{Gr}_n(T\mathcal{F}, \pi)$ ,  $d\eta^j = -\theta_k^j \wedge \eta^k = p_{k\ell}^j \eta^k \wedge \eta^\ell$ . We see that the equation  $d\hat{\alpha} = 0$  is a set of linear conditions on the  $p_{k\ell}^j$ , and that  $V_n(\mathcal{I}, \pi)$  is a submanifold of  $\text{Gr}_n(T\mathcal{F})$ . (The exterior differential system  $\mathcal{I}$  with independence condition  $\eta^1 \wedge \dots \wedge \eta^n \neq 0$  is *in linear form*.) In particular, each  $E \in V_n(\mathcal{I}, \pi)$  is ordinary.

**4.11. Canonical flags and regular presentations.** To each  $n$ -dimensional integral element  $E_n \in V_n(\mathcal{I}, \pi)$  at  $u \in \mathcal{F}$  we may canonically associate a flag  $F = \{E_0 \subset E_1 \subset \dots \subset E_n\}$  by

$$E_k := \{v \in E_n \mid \eta^j(v) = 0 \ \forall j > k\}.$$

The polar spaces are  $H(E_k) = E_n + (\mathfrak{h}_k)_u$ , where the  $\mathfrak{h}_k$  are defined as follows. Let  $i_k : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$  denote the natural inclusion, and set

$$\mathfrak{h}_k := \{x \in \mathfrak{gl}(n) \mid i_k^*(x.\alpha) = 0, \ \forall \alpha \in \Lambda^*(\mathbb{R}^n)^G\}.$$

Note that  $\mathfrak{h}_k$  contains  $\mathfrak{g}$ ,  $\mathfrak{h}_{k+1}$  and the space  $M_{n,k}\mathbb{R}$  of  $n$ -by- $n$  matrices whose first  $k$  columns are zero. When  $G$  is admissible,  $\mathfrak{h}_n = \mathfrak{g}$ .

Cartan's Test implies that

$$\sum_{j=0}^{n-1} c_j = \sum_{j=0}^{n-1} \text{codim}(\mathfrak{h}_j, \mathfrak{gl}(n)) \leq \text{codim}(V_n(\mathcal{I}, \pi), \text{Gr}_n(T\mathcal{F})).$$

A strongly admissible group  $G$  is *regularly presented* when equality holds. We will see that both  $G_2$  and  $Spin(7)$  are regularly presented. When  $G$  is regularly presented every  $E \in V_n(\mathcal{I}, \pi)$  is the terminus of a regular flag  $F$ . It follows from (4.2, 4.3) and Cartan's Test that every  $\overline{E}_n \in V_n(\mathcal{I}, \overline{\pi})$  is also the terminus of a regular flag,  $\{\overline{E}_j := \rho_*(E_j)\}$ .

## 5. REAL ALGEBRAIC AND REAL ANALYTIC STRUCTURES

In this section we briefly review the basics of real algebraic sets. For more information see [1, 2, 3].

A real algebraic set is the set of solutions of polynomial equations in real variables, a set  $V$  of the form  $V(I) = \{x \in \mathbb{R}^n \mid p(x) = 0, \text{ for all } p \in I\}$  where  $I$  is a set of polynomial functions  $p : \mathbb{R}^n \rightarrow \mathbb{R}$ .

A point  $x$  in an algebraic set  $V \subset \mathbb{R}^n$  is called *nonsingular of codimension  $k$  in  $V$*  if there are polynomials  $p_i$ ,  $i \in \{1, \dots, k\}$ , and a neighborhood  $U \subset \mathbb{R}^n$  of  $x$  so that  $p_i(V) = 0$  and

- (i)  $V \cap U = U \cap \bigcap_{i=1}^k p_i^{-1}(0)$
- (ii) the gradients  $\nabla p_i$ , are linearly independent on  $U$ .

Define  $\dim V$  to be the maximum of  $n - k$  over all nonsingular  $x \in V$ . Then for an algebraic set  $V$ ,

$$\text{Nonsing}(V) := \{x \in V \mid x \text{ is nonsingular of dimension } \dim V\}$$

and  $\text{Sing}(V) := V \setminus \text{Nonsing}(V)$ . We say that an algebraic set  $V$  is *nonsingular* if  $\text{Sing}(V) = \emptyset$ .

Nash [20] proved that every smooth, closed manifold is a topological component of a nonsingular algebraic set, and conjectured that every smooth, closed manifold is a nonsingular algebraic set. Tognoli verified the Nash's conjecture.

**Theorem 5.1** (Nash-Tognoli [20, 21, 3]). *Let  $M$  be a smooth, closed manifold. Then there exists a nonsingular algebraic set  $V$  and a diffeomorphism  $\phi : M \rightarrow V$ .*

In [1, 2, 3], Akbulut and King generalized Nash's theorem to interiors of compact manifolds and proved that the interior of a smooth, compact manifold  $M$  is diffeomorphic to a nonsingular real algebraic set  $V$  which is properly imbedded in  $\mathbb{R}^n$  for some  $n$ . This established a one-to-one correspondence between interiors of compact smooth manifolds and nonsingular real algebraic sets.

Note that, one can use Akbulut-King's result and the Real Analytic Implicit Function Theorem, Theorem 5.4, to find real analytic metrics on interiors of compact manifolds. In this paper, we won't use this fact as we first double our manifold to obtain a closed manifold, construct the real analytic metric and then take its restriction.

In [5], Bochner proved that on a closed, real analytic manifold the real analytic differential forms are dense in the smooth forms in the uniform topology. Next, we show that one can obtain Bochner's result using real algebraic theory developed by Akbulut and King.

**Theorem 5.2.** *Every smooth, closed manifold  $X$  can be made a nonsingular real algebraic variety  $V$ . Every smooth differential form on  $V$  can be approximated by a real analytic differential form.*

*Proof.* Let  $X$  be a smooth, closed manifold. By Nash-Tognoli, it can be made a nonsingular real algebraic variety  $V$ . Now, we show that every smooth differential form on  $V$  can be approximated by a real analytic differential form. In [3], it was shown that for a nonsingular algebraic set  $V$  with dimension  $k$ , the classifying Gauss map  $\rho : V \rightarrow G(k, n)$  of the tangent bundle  $TV \rightarrow V$  is an entire rational map. Now, let  $E(k, n)$  be the universal bundle over the Grassmannian variety of  $k$  planes in  $\mathbb{R}^n$ :

$$\begin{aligned} E(k, n) &= \{(A, v) \in \mathcal{M}_{\mathbb{R}}(n) \times \mathbb{R}^n \mid A \in G(k, n), Av = v\} \\ &\downarrow \pi \\ G(k, n) &= \{A \in \mathcal{M}_{\mathbb{R}}(n) \mid A^t = A, A^2 = A, \text{trace}(A) = k\} \end{aligned}$$

where  $\mathcal{M}_{\mathbb{R}}(n)$  denotes  $n \times n$  real matrices. The tangent bundle  $TV$  can be identified with the pullback bundle  $\rho^*E = \{(x, v) \in V \times E \mid \rho(x) = \pi(v)\}$  where  $\rho$  and  $\pi$  are both algebraic.

Then we have the following diagram:

$$\begin{array}{ccc} TV & & E(k, n) \\ \downarrow & & \downarrow \pi \\ V & \longrightarrow & G(k, n) \end{array}$$

where  $V, TV$  are nonsingular algebraic sets. Sections  $s : V \rightarrow TV \cong \rho^*E$  are given by  $s(x) = (x, f(x))$ , for some function  $f : V \rightarrow E$ . This means that for real analytic approximation of the sections  $s$ , it is sufficient to find real analytic approximations of  $f$ .

Now,  $V \subset \mathbb{R}^\ell$  and  $E(k, n) \subset \mathbb{R}^m$  are both nonsingular algebraic sets for some  $\ell, m$ . By the Weierstrass Approximation Theorem, a real valued smooth map from an open subset of  $\mathbb{R}^\ell$  can always be approximated by polynomials. Denote this polynomial approximating  $f$  as  $F$ . Even though  $F$  may not map  $V$  to  $E$ , we can always take the projection  $\pi$  from the image of  $F$  to  $E$ . This map, which is from a tubular neighborhood of an algebraic variety  $E$  to  $E$ , is real analytic. So the composition of  $F$  and  $\pi : \text{tubular}(E) \subset \mathbb{R}^N \rightarrow E$  yields a real analytic approximation of  $f$ , and thus the section  $s = (x, f(x))$ .

Since the cotangent and tangent bundles are the dual bundles, the same real analytic approximation holds for differential forms.  $\square$

An important property of real analytic functions is that the inverse of a real analytic function is also real analytic.

**Theorem 5.3** (Real analytic inverse function theorem). *Let  $F$  be real analytic in a neighborhood of  $a = (a_1, a_2, \dots, a_n)$  and suppose that its derivative at  $a$ ,  $DF(a)$ , is nonsingular. Then  $F^{-1}$  is defined and real analytic in a neighborhood of  $F(a)$ .*

The proof of this theorem follows from a special case of the Cauchy-Kowalewsky Theorem [16, 17].

As an important corollary, we obtain the implicit function theorem in the analytic setting.

**Theorem 5.4** (Real analytic implicit function theorem). *Suppose  $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$  is real analytic in a neighborhood of  $(x_0, y_0)$ , for some  $x_0 \in \mathbb{R}^n$  and some  $y_0 \in \mathbb{R}^m$ . If  $F(x_0, y_0) = 0$  and the  $m \times m$  matrix with entries  $\frac{\partial F_i}{\partial y_j}(x_0, y_0)$  is nonsingular, then there exists a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  which is real analytic in a neighborhood of  $x_0$  and is such that  $F(x, f(x)) = 0$  holds in a neighborhood of  $x_0$ .*

Assume that  $V \subset \mathbb{R}^n$  is an algebraic set, and define  $Z = \text{Nonsing}(V)$ . Let  $g_0$  denote the canonical Euclidean metric on  $\mathbb{R}^n$ . The Implicit Function Theorem for real analytic maps [16, 17] implies that the restriction of  $g_0$  to  $Z$  is a real analytic Riemannian metric.

Let  $K$  be a smooth, compact manifold. Then the double  $\text{doub}(K)$  is smooth and closed. From Theorem 5.1 and Theorem 5.4, we deduce the following lemma.

**Lemma 5.5.** *Let  $\text{doub}(K)$  be the double of a smooth, compact manifold  $K$ . Then  $\text{doub}(K)$  admits a compatible real analytic structure and a real analytic metric  $g$ .*

## 6. THE ASSOCIATIVE EMBEDDING: PROOF OF THEOREMS 1.1 & 1.3

Bryant has shown that the group  $G_2$  is admissible [7, Prop.1]. Now consider the differential system  $\mathcal{I}$  of §4.8. Here  $n = 7$  and the indices  $j, k$  range over  $1, \dots, 7$ . On  $\mathcal{F}$ , the ideal is generated by the 4-form  $d\widehat{\varphi}_0$  and the 5-form  $d(*\widehat{\varphi}_0)$  where

$$\begin{aligned} \widehat{\varphi}_0 &= \eta^{123} + \eta^1 \wedge (\eta^{45} + \eta^{67}) + \eta^2 \wedge (\eta^{46} - \eta^{57}) + \eta^3 \wedge (-\eta^{47} - \eta^{56}) \\ * \widehat{\varphi}_0 &= \eta^{4567} + \eta^{23} \wedge (\eta^{45} + \eta^{67}) + \eta^{31} \wedge (\eta^{46} - \eta^{57}) + \eta^{12} \wedge (-\eta^{47} - \eta^{56}). \end{aligned}$$

Given a  $\mathfrak{gl}(7)$ -valued connection form  $\theta_k^j$  on  $\mathcal{F}$ , we have  $d\eta^j = -\theta_k^j \wedge \eta^k$ . Let  $p_{k\ell}^j(u)$  be the functions parameterizing  $\text{Gr}_7(T_u\mathcal{F}, \pi)$  introduced in §4.10: on any  $E \in \text{Gr}_7(T_u\mathcal{F}, \pi)$  we have  $\theta_k^j = p_{k\ell}^j \eta^\ell$ . Then the equations  $d\widehat{\varphi}_0 = 0 = d(*\widehat{\varphi}_0)$  defining  $V_7(\mathcal{I}, \pi)$  in  $\text{Gr}_7(T\mathcal{F})$  are linear conditions on the parameters  $p_{k\ell}^j$ . The

first equation is equivalent to 35 (independent) linear constraints on the  $p_{k\ell}^j$ , and the second equation imposes an additional 14. Hence

$$\text{codim}(V_7(\mathcal{I}, \pi), \text{Gr}_7(\mathcal{F})) = 49.$$

Moreover, by work of Fernandez and Gray [11] we know that  $\dim h^0(\mathfrak{g}_2) = 49$ . Whence  $G_2$  is strongly admissible (§4.9).

Fix  $E \in V_7(\mathcal{I}, \pi)$ , and let  $F$  denote the canonical flag of §4.11. Next we compute the sequence of polar spaces. Since  $(\Lambda^* \mathbb{R}^7)^{G_2}$  contains no 1- or 2-forms, we have  $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathfrak{h}_2 = M_7 \mathbb{R} \simeq \mathbb{R}^{49}$ . Next,  $i_3^*(x.\varphi_0) = (x_1^1 + x_2^2 + x_3^3) dx^1 \wedge dx^2 \wedge dx^3$ , so that  $\mathfrak{h}_3 = \{x \in M_7 \mathbb{R} \mid x_1^1 + x_2^2 + x_3^3 = 0\}$ . Similarly,  $i_4^*(x.\varphi_0) = 0 = i_4^*(x.*\varphi_0)$  implies that  $\mathfrak{h}_4 \subset \mathfrak{h}_3$  is given by the four additional equations

$$0 = x_4^1 - x_3^6 - x_2^7 = x_4^2 + x_3^5 + x_1^7 = x_4^3 - x_2^5 + x_1^6 = x_1^5 + x_2^6 - x_3^7.$$

Continuing in this fashion we find that  $\text{codim}(\mathfrak{h}_5) = 15$ ,  $\text{codim}(\mathfrak{h}_6) = 13$ , and  $\mathfrak{h}_7 = \mathfrak{g}_2$  so that  $\text{codim}(\mathfrak{h}_7) = 35$ .

Whence the polar space codimensions are  $(c_0, c_1, \dots, c_6) = (0, 0, 0, 1, 5, 15, 28)$  and  $\sum c_j = 49$ . In particular,  $G_2 \subset \text{SO}(7)$  is regularly presented (§4.11). Additionally, Cartan's Test implies that  $V_7(\mathcal{I}, \pi)$  is a codimension 49 submanifold of  $\text{Gr}(\mathcal{F}, \pi)$ , and each  $E \in V_7(\mathcal{I}, \pi)$  is the terminus of a (canonical) regular flag  $F$ .

This completes the necessary preliminaries for  $G_2$ . For Theorem 1.1 we assume that  $(K^3, g)$  is a closed, oriented real analytic Riemannian 3-manifold. As an oriented 3-manifold  $K$  is smoothly parallelizable by a result of Wu [19]. Using Bochner's result, [5], or Theorem 5.2, we can conclude that  $K$  admits a real analytic parallelization.

In the case of Theorem 1.3, invoke Lemma 5.5 to endow  $\text{doub}(K)$  with a compatible real analytic Riemannian structure  $(\text{doub}(K), g)$ . (If  $K$  is closed, then  $\text{doub}(K) = K$ .) As above,  $\text{doub}(K)$  admits a real analytic parallelization.

The rest of the argument applies to both theorems. Let  $A = \text{int}(K)$ . (In Theorem 1.1,  $A = K$ .) Assume  $A$  is connected, else apply the theorem to each connected component individually. Restrict the Riemannian metric  $g$  and real analytic parallelization to  $A$ . The Gramm-Schmidt process yields an orthonormal parallelization, and 1-forms  $\omega_1, \omega_2$  and  $\omega_3$  such that

$$g = \omega_1^2 + \omega_2^2 + \omega_3^2,$$

and  $\text{dvol}_g = \omega_1 \wedge \omega_2 \wedge \omega_3$ .

Let  $M = A \times \mathbb{R}^4$ , and let  $y = (y^4, y^5, y^6, y^7)$  be linear coordinates on  $\mathbb{R}^4$ . Regard the  $y^j$  as functions on  $M$  and identify  $A$  with the 0-section  $A \times \{0\}$ . The 1-forms  $\{\omega, dy\}$  form a coframing of  $M$  and define a global section  $s : M \rightarrow \mathcal{F}$ . The corresponding trivialization of  $\pi : \mathcal{F} \rightarrow M$  is given by associating to each  $u \in \mathcal{F}_z$  the unique  $g = g(u) \in \text{GL}(7)$  such that  $u = g^{-1} \circ s(z)$ . With respect

to the trivialization, the canonical 1-forms  $\eta = (\eta^j)$  are given by

$$\eta_{(z,g)}(v) = g^{-1} \left( \begin{array}{c} \omega(\pi_* v) \\ dy(\pi_* v) \end{array} \right), \quad v \in T_{z,g}(M \times \mathrm{GL}(7)).$$

It will be convenient notationally to identify  $\mathcal{F}$  with the trivialization  $M \times \mathrm{GL}(7)$ .

Define an involution  $r : M \rightarrow M$  by  $r(p, y) = (p, -y)$ ,  $(p, y) \in A \times \mathbb{R}^4 = M$ . Lift  $r$  to an involution, also denoted by  $r$ , of  $\mathcal{F}$  by defining  $r(u) = r^*(u)$ . That is, given  $u : T_z M \rightarrow \mathbb{R}^7$  in  $\mathcal{F}_z$  we define  $r(u) : T_{r(z)} M \rightarrow \mathbb{R}^7$  in  $\mathcal{F}_{r(z)}$  to be the map sending  $v \in T_{r(z)} M$  to  $u(r_* v)$ . With respect to the trivialization  $\mathcal{F} \simeq A \times \mathbb{R}^4 \times \mathrm{GL}(7)$ , we have  $r(p, y, g) = (p, -y, Rg)$ , where

$$R = \begin{pmatrix} I_3 & 0 \\ 0 & -I_4 \end{pmatrix}.$$

Notice that  $R \in G_2$ , so that  $r : \mathcal{F} \rightarrow \mathcal{F}$  preserves the  $\rho$ -fibres, and  $r$  descends to a well-defined involution of  $S$ . Also,  $\pi \circ r = r \circ \pi$  implies that  $r^* \eta = \eta$ , so that

$$r^*(\widehat{\varphi_0}) = \widehat{\varphi_0} \quad \text{and} \quad r^*(\widehat{* \varphi_0}) = \widehat{* \varphi_0}.$$

Whence  $r^* \mathcal{I} = \mathcal{I}$ , and  $r$  carries integral manifolds of  $\mathcal{I}$  to integral manifolds of  $\mathcal{I}$ .

We are now ready to apply the Cartan-Kähler theorem to prove Theorem 1.1. Define a lift  $f_3 : A \rightarrow S$  by  $f_3(p, 0) = \rho(p, 0, I_7)$ , and let  $X_3$  denote the image. Since  $X_3$  is three-dimensional, and  $\mathcal{I}$  is generated by a 4- and 5-form,  $X_3$  is trivially an integral manifold. Because  $R \in G_2$ ,  $X_3$  lies in the fixed locus of  $r$ . We will use the Cartan-Kähler theorem to thicken (in four steps)  $X_3$  to a seven-dimensional  $r$ -invariant integral manifold that projects diffeomorphically onto a neighborhood  $N$  of  $A \subset M$ . Moreover, the induced  $G_2$ -structure on  $N$  will have the properties that (i)  $A \subset N$  is associative, and (ii) the metric induced on  $N$  by the  $G_2$ -structure agrees with  $g$  when restricted to  $A$ . The construction is repetitive and very similar to that of [8], so after detailing Steps 1 and 2 below, we will sketch the remaining steps.

*Step 1: Thicken  $X_3$  to a 4-dimensional integral manifold  $X_4$  of  $\mathcal{I}$ .* Let  $z = f_3(p)$ . To compute the polar space  $H(T_z X_3)$ , note that  $T_z X_3 = \rho_* T_p$ , where  $T_p$  is the 3-plane tangent to the lift  $\{(p, 0, I_7) \mid p \in A\}$  in  $\mathcal{F}$ . There exists an  $E_7 \in V_7(\mathcal{I}, \pi)$  containing  $T_p$ . Given this, it is clear that  $T_p = E_3$  in the canonical regular flag  $F$  terminating in  $E_7$ . Thus  $T_z X_3$  is  $\overline{E}_3$  in the canonical regular flag  $\overline{F}$  (cf. §4.11).

To see that such an  $E_7$  exists, recall that  $(p_{k\ell}^j) \in \mathbb{R}^{7^3}$  parameterizes the open set of  $\pi$ -transverse  $E \in \mathrm{Gr}_7(T_u \mathcal{F}, \pi)$ , and  $V_7(\mathcal{I}, \pi)$  is a linear subspace of  $\mathbb{R}^{7^3}$  of codimension 49. The condition that  $T_p^3$  lie in some  $E_7$  holds if we are free to specify the values of  $p_{k1}^j, p_{k2}^j, p_{k3}^j$  in  $V_7(\mathcal{I}, \pi)$ . It can be checked that



these variables are independent in  $V_7(\mathcal{I}, \pi) \subset \mathbb{R}^{7^3}$ , so such a specification is always possible.

We may now compute  $\dim H(T_{f_3(p)}X_3) = \dim H(\overline{E}_3) = 41$ . Hence  $X_3$  is a regular integral manifold of extension rank 37. Note that

$$\dim S = 42;$$

so to apply the Cartan-Kähler Theorem we need to construct a 5-dimensional manifold  $Z_3$  that contains  $X_3$  with tangent space at  $z \in X_3$  transverse to  $H(T_zX_3)$ .

Let  $W_1 \subset M_7\mathbb{R}$  be the 1-dimensional subspace of matrices of the form

$$\begin{pmatrix} x_1 I_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_1 \in \mathbb{R}.$$

Notice that  $W_1 \cap \mathfrak{h}_3 = \{0\}$  and  $RW_1 = W_1$ . Since  $\mathfrak{g}_2 \subset \mathfrak{h}_3$ , the affine space  $I_7 + W_1$  intersects  $G_2$  transversely at  $I_7 \in \text{GL}(7)$ . Hence, there is a neighborhood  $U_1$  of 0 in  $W_1$  such that the map  $U_1 \rightarrow \text{GL}(7)/G_2$  sending  $x \mapsto (I + x)G_2$  is an embedding.

Define a 5-dimensional manifold  $Z_3 \subset S$  by

$$Z_3 := \{ \rho(p, (y^4, 0, 0, 0), I_7 + x) \mid p \in A, y^4 \in \mathbb{R}, x \in U_1 \}.$$

As constructed  $Z_3$  contains  $X_3$ , is  $r$ -invariant and  $H(T_zX_3) \cap T_zZ_3$ ,  $z \in X_3$ , is of dimension 4. The Cartan-Kähler theorem concludes that there exists a real analytic, 4-dimensional integral manifold  $Y_4 \subset Z_3$  containing  $X_3$ . Since  $r^*\mathcal{I} = \mathcal{I}$ ,  $r(Y_4)$  is also an integral manifold. And since  $X_3$  and  $Z_3$  are  $r$ -invariant, we have  $X_3 \subset r(Y_4) \subset Z_3$ . By the uniqueness part of the Cartan-Kähler theorem the  $r$ -invariant  $X_4 := Y_4 \cap r(Y_4)$  is also a 4-dimensional integral manifold of  $\mathcal{I}$ .

Given  $z \in X_3$  note that the 4-plane  $T_zX_4 = H(T_zX_3) \cap T_zZ_3$  is (i)  $\pi$ -transverse, and (ii) the  $\overline{E}_4$  of a regular flag  $\overline{F}$ . Transversality implies that a neighborhood of  $X_3$  in  $X_4$  projects diffeomorphically onto a neighborhood  $N_4$  of  $A$  in  $A \times \mathbb{R} \subset M$ . Shrinking  $X_4$  if necessary, we may assume that  $X_4$  is image of a section  $N_4 \rightarrow S$ . Item (ii) implies that  $T_zX_4$  is regular. Since regularity is an open condition, again shrinking  $X_4$  if necessary, we may assume that  $X_4$  is regular. Finally, we may suppose (shrinking again if necessary) that  $X_4$  is connected. Whence the extension rank of  $X_4$  is 32.

*Step 2: Thicken  $X_4$  to a 5-dimensional integral manifold  $X_5$ .* To apply the Cartan-Kähler theorem we must construct a 10-dimensional manifold  $Z_4$  containing  $X_4$  so that  $T_zZ_4$  and  $H(T_zX_4)$  are transverse along  $X_4$ . Define  $W_5 \subset M_7\mathbb{R}$  to be the 5-dimensional subspace

$$\begin{pmatrix} x_1 & 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_5 & x_3 & x_3 & 0 & 0 & 0 & 0 \\ x_4 & x_5 & x_3 & 0 & 0 & 0 & 0 \\ x_4 & x_4 & x_5 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5.$$

Notice that  $W_1 \subset W_5$ ,  $W_5 \cap \mathfrak{h}_4 = \{0\}$ , and  $RW_5 = W_5$ . Since  $\mathfrak{g}_2 \subset \mathfrak{h}_4$ , the affine space  $I_7 + W_5$  intersects  $G_2$  transversely at  $I_7 \in \mathrm{GL}(7)$ . Hence there is a neighborhood  $U_5$  of  $0 \in W_5$  such that the map  $U_5 \rightarrow \mathrm{GL}(7)/G_2$  sending  $x \rightarrow (I + x)G_2$  is an embedding.

Define a 10-dimensional manifold  $Z_4 \subset S$  by

$$Z_4 := \{ \rho(p, (y^4, y^5, 0, 0), I_7 + x) \mid p \in A, (y^4, y^5) \in \mathbb{R}^2, x \in U_5 \}.$$

By construction  $Z_4$  contains  $X_4$ , is  $r$ -invariant, and the intersection  $H(T_z X_4) \cap T_z Z_4$ ,  $z \in X_4$ , is of dimension five. Thus, the Cartan-Kähler theorem yields a 5-dimensional, real analytic integral manifold  $Y^5$  such that  $X_4 \subset Y^5 \subset Z_4$ . Since  $\mathcal{I}$  is preserved under  $r$ ,  $r(Y^5)$  is also an integral manifold. The  $r$ -invariance of  $X_4$  and  $Z_4$  implies  $X_4 \subset r(Y^5) \subset Z_4$ . The uniqueness portion of the Cartan-Kähler theorem assures us that  $X_5 := Y^5 \cap r(Y^5)$  is also a 5-dimensional, real analytic integral manifold of  $\mathcal{I}$ .

Given  $z \in X_3$ ,  $T_z X_5 = H(T_z X_4) \cap T_z Z_4$  is (i)  $\bar{\pi}$ -transverse, and (ii) the  $\bar{E}_5$  of a canonical regular flag. Transversality implies that a neighborhood of  $X_3$  in  $X_5$  projects diffeomorphically onto a neighborhood  $N_5$  of  $A$  in  $A \times \mathbb{R}^2 \subset M$ . So, shrinking  $X_5$  if necessary, we may assume that it is the (connected) image of a section  $N_5 \rightarrow S$ . Moreover, since regularity is an open condition, a neighborhood of  $X_3$  in  $X_5$  will be regular. Hence, again shrinking  $X_5$  if necessary, we may take  $X_5$  to be a regular integral manifold. The extension rank of  $X_5$  is 21.

*Steps 3 & 4.* As in Steps 1 & 2 we may thicken  $X_5$  to a 6-dimensional integral manifold  $X_6$  of extension rank 7. Then  $X_6$  is thickened to a 7-dimensional integral manifold  $X_7$  that is an  $r$ -invariant connected image of a section  $\sigma : N \rightarrow S$  over an open neighborhood  $N$  of  $A$  in  $M$ .

*The finish.* As a section  $N \rightarrow S$ ,  $\sigma$  represents a  $G_2$ -structure on the 7-dimensional  $N$ . The corresponding 3-form on  $N$  is  $\varphi := \sigma^*(\widehat{\varphi}_0)$ . By construction  $\sigma(N) \subset S$  is an integral manifold of  $\mathcal{I}$ . Equivalently, the  $G_2$ -structure is torsion-free, and  $(N, \varphi)$  is a  $G_2$ -manifold.

The relation  $r \circ \sigma = \sigma \circ r$  implies that  $r : M \rightarrow M$  restricts to an involution on  $N$ , and that  $r^* \varphi = \varphi$ . Whence  $r$  is a  $G_2$ -involution (§2.2), Lemma 2.3. It follows immediately that  $A$ , as the fixed point locus of  $r$  in  $N$ , is associative.

Let  $h$  denote the metric induced on  $N$  by the  $G_2$ -structure. At  $z \in A$ ,  $\{\omega_1, \omega_2, \omega_3, dy^4, dy^5, dy^6, dy^7\}$  is a  $G_2$  coframing of  $T_z^*N$ . In particular,

$$\begin{aligned}\varphi_z &= \omega_{123} + \omega_1 \wedge (dy^{45} + dy^{67}) + \omega_2 \wedge (dy^{46} - dy^{57}) \\ &\quad + \omega_3 \wedge (-dy^{47} - dy^{56}), \\ h_z &= \omega_1^2 + \omega_2^2 + \omega_3^2 + (dy^4)^2 + (dy^5)^2 + (dy^6)^2 + (dy^7)^2.\end{aligned}$$

Whence  $h|_A = g$ , and the inclusion  $i : A \hookrightarrow N$  is an isometry. (Also,  $i^*\varphi = \omega_{123} = \text{dvol}_A$ , proving again that  $A$  is associative.) This completes the proof of Theorem 1.1.

*Remark.* We remarked in §1 after the statement of Theorem 1.1 that, so long as  $A$  is not flat,  $\text{Hol}(N) = G_2$ . In particular,  $N \neq CY \times \mathbb{R}$ . This may be seen as follows. Suppose that  $\text{Hol}(N) \neq G_2$ . Then  $\text{Hol}(N) \subseteq \text{SU}(3)$  and  $N$  is an  $\text{SU}(3)$ -manifold.

Set  $\mathbf{i} = \sqrt{-1}$  and take the  $\text{SU}(3)$  action on  $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}$  that fixes the forms  $dx^7$ ,

$$\begin{aligned}\omega_0 &= dx^{16} - dx^{25} - dx^{34}, \\ \Upsilon_0 &= (dx^1 + \mathbf{i} dx^6) \wedge (dx^2 - \mathbf{i} dx^5) \wedge (dx^3 - \mathbf{i} dx^4).\end{aligned}$$

Then  $\text{SU}(3)$  acts trivially on the second factor and by the standard representation on the first.

Let  $R = \mathcal{F}/\text{SU}(3)$  and consider

$$\begin{array}{ccc}\mathcal{F} & \xrightarrow{\mu} & R \\ \pi \downarrow & & \downarrow \nu \\ N & \xleftarrow{\tilde{\pi}} & S\end{array}$$

Above,  $\nu \circ \mu = \rho$ . Let  $\tilde{\pi} = \tilde{\pi} \circ \nu : R \rightarrow N$ .

If  $\text{Hol}(N) \subseteq \text{SU}(3)$ , then  $N$  is an  $\text{SU}(3)$ -manifold. In particular,  $N$  admits a section  $\tau : N \rightarrow R$  such that  $\nu \circ \tau = \sigma$ . Rename  $\mathcal{I} = \mathcal{I}_{G_2}$ , and let  $\mathcal{I}_{\text{SU}(3)}$  be the ideal generated by  $d\eta^7$ ,  $d\omega$  and  $d\Upsilon$ , where

$$\begin{aligned}\omega &= \eta^{16} - \eta^{25} - \eta^{34} \\ \Upsilon &= (\eta^1 + \mathbf{i} \eta^6) \wedge (\eta^2 - \mathbf{i} \eta^5) \wedge (\eta^3 - \mathbf{i} \eta^4).\end{aligned}$$

The image  $\tau(N)$  is necessarily an integral manifold of  $\mathcal{I}_{\text{SU}(3)}$ .

Fix  $z \in X_3$ , and let  $E \in V_7(\mathcal{I}_{G_2}, \pi)$  denote the 7-plane constructed in the proof above with  $\rho_*E = T_z X_7 = T_z \sigma(N) \in V_7(\mathcal{I}_{G_2}, \tilde{\pi})$ . If  $\tau : N \rightarrow R$  exists, then there exists  $E' \in V_7(\mathcal{I}_{\text{SU}(3)}, \pi) \subset V_7(\mathcal{I}_{G_2}, \pi)$  such that  $\mu_*E' \in V_7(\mathcal{I}_{\text{SU}(3)}, \tilde{\pi})$  is tangent to  $\tau(N)$  and  $\rho_*E' = \nu_*(\mu_*E') = \rho_*E$ . As noted in §4.10, this implies  $E' \equiv E \pmod{\mathfrak{g}_2}$ . A lengthy computation confirms that this is possible if and only if  $A$  is flat.

## 7. THE CAYLEY EMBEDDING: PROOF OF THEOREMS 1.2 &amp; 1.4

*Remark.* As an application of the Cartan-Kähler Theorem the proof of Theorems 1.2 and 1.4 is very like the proof of Theorems 1.1 and 1.3. However, unlike 3-manifolds, the 4-manifold  $A = \text{int}(K)$  may not admit a global parallelism. We assume that the bundle of self-dual 2-forms on  $\text{doub}(K)$  is trivial in order to obtain the structure necessary to apply the Cartan-Kähler Theorem.

The group  $\text{Spin}(7)$  is admissible [7];  $\Lambda^*(\mathbb{R}^8)^{\text{Spin}(7)}$  is generated by the 4-form  $\Psi_0$ . The differential system  $\mathcal{I}$  of §4.8 is generated by the exterior derivative of

$$\begin{aligned} \widehat{\Psi}_0 = & \eta^{0123} + \eta^{4567} + (\eta^{01} + \eta^{23}) \wedge (\eta^{45} + \eta^{67}) \\ & + (\eta^{02} + \eta^{31}) \wedge (\eta^{46} + \eta^{75}) + (\eta^{03} + \eta^{12}) \wedge (\eta^{74} + \eta^{65}) . \end{aligned}$$

The condition that  $d\widehat{\Psi}_0 = 0$  is 56 independent linear equations on the functions  $p_{k\ell}^j(u)$  parameterizing  $\text{Gr}_8(T_u\mathcal{F}, \pi)$  (cf. §4.10). Thus

$$\text{codim}(V_8(\mathcal{I}, \pi), \text{Gr}_8(T\mathcal{F})) = 56 .$$

Since  $\dim \mathfrak{h}^0(\mathfrak{spin}(7)) = 56$  [7, Prop.4], it follows that  $\text{Spin}(7)$  is strongly admissible, c.f. §4.9.

Fix  $E \in V_8(I, \pi)$ , and let  $F$  denote the canonical flag of §4.11. The subspaces  $\mathfrak{h}_k$  of §4.11 are:  $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}_3 = M_8\mathbb{R} \simeq \mathbb{R}^{64}$ . The subspace  $\mathfrak{h}_4 \subset M_8\mathbb{R}$  is defined by the single equation  $x_0^0 + x_1^1 + x_2^2 + x_3^3 = 0$ . As in §6 calculations of the remaining  $\mathfrak{h}_j$  lead us to the polar space codimensions:  $(c_0, c_1, \dots, c_8) = (0, 0, 0, 0, 1, 5, 15, 35, 43)$  and  $\sum c_j = 56$ , so that  $\text{Spin}(7)$  is regularly presented, c.f. §4.11. Cartan's Test concludes that  $V_8(\mathcal{I}, \pi)$  is a codimension 56 (observed above) submanifold of  $\text{Gr}_8(T\mathcal{F}, \pi)$ , and each  $E \in V_8(\mathcal{I}, \pi)$  is the terminus of a canonical regular flag  $F$ .

This completes the necessary preliminaries for  $\text{Spin}(7)$ . For Theorem 1.2 we assume that  $(K^4, g)$  is a closed oriented, real analytic Riemannian 4-manifold, and that the bundle  $\Lambda_+^2(K)$  of self-dual 2-forms over  $K$  is smoothly trivial. Bochner's result or Theorem 5.2 implies that  $\Lambda_+^2(K)$  is real-analytically trivial. In particular, there exist globally defined real-analytic self-dual 2-forms  $\Omega_1, \Omega_2, \Omega_3$  such that  $\Omega_j \wedge \Omega_k = 2\delta_{jk} \text{dvol}_g$ .

In the case of Theorem 1.4, invoke Lemma 5.5 to endow  $\text{doub}(K)$  with a compatible real analytic Riemannian structure  $(\text{doub}(K), g)$ . (If  $K$  is closed, then  $\text{doub}(K) = K$ .) As above,  $\text{doub}(K)$  admits globally defined real-analytic self-dual 2-forms  $\Omega_1, \Omega_2, \Omega_3$  such that  $\Omega_j \wedge \Omega_k = 2\delta_{jk} \text{dvol}_g$ .

The rest of the argument applies to both theorems. Let  $A = \text{int}(K)$ . (In Theorem 1.2,  $A = K$ .) Assume  $A$  is connected, else apply the theorem to each connected component individually. Restrict the Riemannian metric  $g$  and 2-forms  $\Omega_j$  to  $A$ . While  $A$  may not admit a global coframing, it is not difficult to check that there exist local orthonormal coframings  $\{\omega^a\}_{a=1}^4$  such that

$$\Omega_1 = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4, \quad \Omega_2 = \omega^1 \wedge \omega^3 - \omega^2 \wedge \omega^4, \quad \Omega_3 = \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3 .$$

This choice of coframing is unique up to the action of  $SU(2)$ . (Recall,  $SU(2)$  is the subgroup of  $SO(4)$  preserving the two forms  $\frac{i}{2}(\zeta^1 \wedge \bar{\zeta}^1 + \zeta^2 \wedge \bar{\zeta}^2)$  and  $\zeta^1 \wedge \zeta^2$ , where  $\zeta^1 = \omega^1 + i\omega^2$  and  $\zeta^2 = \omega^3 + i\omega^4$ .) We may identify  $SU(2)$  with the subgroup of  $Spin(7)$  fixing  $dx^j$ ,  $j = 0, \dots, 3$ , via

$$\left\{ \begin{pmatrix} \text{Id}_4 & 0 \\ 0 & P \end{pmatrix} \middle| P \in SU(2) \right\} \subset Spin(7).$$

Let  $M = \mathbb{R}^4 \times A$  with linear coordinates  $\{y^j\}_{j=0}^3$  on  $\mathbb{R}^4$ . Then

$\Psi = dy^{0123} + \frac{1}{2}\Omega_1 \wedge \Omega_1 + (dy^{01} + dy^{23}) \wedge \Omega_1 + (dy^{02} + dy^{31}) \wedge \Omega_2 - (dy^{03} + dy^{12}) \wedge \Omega_3$  defines a  $Spin(7)$ -structure on  $M$ . Let  $\sigma : M \rightarrow S$  denote the corresponding section.

Define an involution on  $M$  by  $r(y, p) = (-y, p)$ . Note that  $r^*\Psi = \Psi$ . Define a covering involution  $r : \mathcal{F} \rightarrow \mathcal{F}$  as follows. Given  $u : T_z M \rightarrow \mathbb{R}^8$ , let  $\widehat{r}(u)$  be the coframe  $r^*(u) : T_{r(z)} M \rightarrow \mathbb{R}^8$ . Then  $r^*(\eta) = \eta$  and  $r^*\widehat{\Psi}_0 = \widehat{\Psi}_0$ . Let  $\{\omega^a\}$  be a coframing of an open set  $U \subset L$ . Then  $\{dy^j, \omega^a\}$  defines a trivialization  $\mathcal{F}|_{\mathbb{R}^4 \times U} := \pi^{-1}(\mathbb{R}^4 \times U) \simeq \mathbb{R}^4 \times U \times GL_8\mathbb{R}$ . Notice that  $\mathcal{F}|_{\mathbb{R}^4 \times U}$  is invariant under  $r$  and, with respect to the trivialization, is given by  $r(y, p, g) = (-y, p, Rg)$ , where

$$R = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \in Spin(7).$$

In particular,  $r$  descends to a well-defined involution on  $S$ .

From this point on the proof of Theorem 1.2 is very similar to the proof of Theorem 1.1; so we merely sketch the main steps. Define  $X_4 = \sigma(A)$ . Since  $\mathcal{I}$  is generated by a 5-form, and  $X_4$  is 4-dimensional,  $X_4$  is trivially an integral manifold of  $\mathcal{I}$ . Since  $r \circ \sigma = \sigma \circ r$ , it follows that  $X_4$  lies in the fixed point locus of  $r : S \rightarrow S$ .

It remains to select the subspaces  $W_{d_1} \subset W_{d_2} \subset W_{d_3} \subset W_{d_4} \subset M_8\mathbb{R}$ ,  $(d_4, d_5, d_6, d_7) = (1, 5, 15, 35)$ , so that (i)  $\dim W_{d_j} = d_j$ , (ii)  $W_{d_s} \cap \mathfrak{h}_s = \{0\}$ , and (iii)  $RW_d \subset W_d$ . Because the coframing  $\omega, dy$  of  $M$  is defined only up to the  $SU(2)$  action it is also necessary that we pick the subspaces so that  $SU(2)W_d \subset W_d$ . We leave this exercise to the reader.

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